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Quasinormal Modes of a Black Hole with Quadrupole Moment

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We analytically determine the quasinormal mode (QNM) frequencies of a black hole with quadrupole moment in the eikonal limit using the light-ring method. The generalized black holes that are discussed in this work possess arbitrary quadrupole and higher mass moments in addition to mass and angular momentum. In particular, the generalized black hole that we consider for our extensive calculations is a completely collapsed configuration whose exterior gravitational field can be described by the Hartle-Thorne spacetime [Astrophys. J. **153***, 807-834 (1968)]. This collapsed system as well as its QNMs is characterized by mass M, quadrupole moment Q and angular momentum J, where the latter two parameters are treated to first and second orders of approximation, respectively. When the quadrupole moment is set equal to the relativistic quadrupole moment of the corresponding Kerr black hole, $J^2/(Mc^2)$, the Hartle-Thorne QNMs reduce to those of the Kerr black hole to second order in angular momentum J. Using ringdown frequencies, one cannot observationally distinguish a generalized Hartle-Thorne black hole with arbitrary quadrupole moment from a Kerr black hole provided the dimensionless parameter given by $|QMc^2 - J^2|c^2/(G^2M^4)$ is sufficiently small compared to unity.*

We consider that the final state of collapse is characterized by a set of exterior multipole moments of the system. We treat the angular momentum to second order and the quadrupole contribution to first order. We study analytically the QNMs of this configuration in the eikonal limit using the light-ring method. Consider the δ-metric Ref. [1] given by

$$ds^2 = -\mathbb{A}^{\delta}\, dt^2 + \mathbb{A}^{-\delta} \left(\frac{\mathbb{A}}{\mathbb{B}}\right)^{\delta^2-1} dr^2 + \mathbb{A}^{1-\delta} \left(\frac{\mathbb{A}}{\mathbb{B}}\right)^{\delta^2-1} r^2\, d\theta^2 + \mathbb{A}^{1-\delta}\, r^2 \sin^2\theta\, d\phi^2 \qquad (1)$$

$$\delta = 1 + q\,. \qquad (2)$$

We have verified from our more general results that the singularities of the δ-metric are just the ones described in Ref. [1]. All the singularities occur for $r \leq 2m$; for $r > 2m$, the exterior field of the δ-metric is singularity-free and static. For an oblate configuration, the $r = 2m$ hypersurface is null provided $q < (\sqrt{5}-1)/2$. So one can set up ingoing boundary conditions for QNMs in this case just as in the case of a black hole. We can replace δ by $1+q$, m by $M/(1+q)$ and then linearize the resulting metric in the dimensionless quadrupole parameter q. To first order in q, the quadrupole moment is then given by Q, where

$$Q = \frac{2}{3} M^3 \, q \, . \tag{3}$$

The metric is

$$\begin{aligned} ds^2_{SQ} = & -\left[1 + q\left(\frac{2M}{r\mathbb{A}} + \ln \mathbb{A}\right)\right] \mathbb{A} \, dt^2 + \left[1 - q\left(\frac{2M}{r\mathbb{A}} + \ln \frac{\mathbb{B}^2}{\mathbb{A}}\right)\right] \frac{dr^2}{\mathbb{A}} \\ & + \left(1 - q \ln \frac{\mathbb{B}^2}{\mathbb{A}}\right) r^2 \, d\theta^2 + (1 - q \ln \mathbb{A}) \, r^2 \sin^2\theta \, d\phi^2 \, . \end{aligned} \tag{4}$$

We now consider the geodesics in the equatorial plane. We have shown that the last null orbit is given by

$$r_0 = 3M\left(1 - \frac{1}{3} q\right) , \qquad \frac{d\phi}{dt} = \omega_\pm = \pm \frac{1}{3\sqrt{3}\, M} \left[1 - q\left(-1 + \ln 3\right)\right] . \tag{5}$$

To find the quasinormal modes we use the light ring method in which the divergence of the null rays away from the unperturbed orbit corresponds to the decay of the QNM wave amplitude with time in the eikonal limit [2, 3]. The imaginary parts of the QNMs are therefore given by the decay rate of the orbit. Using this method we have shown that

$$\omega_{QNM} = \omega^0_{SQ} + i\Gamma_{SQ} = \pm j\omega_\pm + i\gamma_{SQ}\left(n + \frac{1}{2}\right) , \qquad n = 0, 1, 2, 3, \cdots , \tag{6}$$

where

$$\omega_\pm = \pm \frac{1 - q(\ln 3 - 1)}{3\sqrt{3}M} , \qquad \gamma_{SQ} = \frac{1 + q\left[1 + 2\ln(2/3)\right]}{3\sqrt{3}M} , \qquad q = \frac{3Q}{2M^3} . \tag{7}$$

The other metric we consider is the static Hartle-Thorne metric for a mass M with quadrupole moment Q, the latter treated to linear order [4]

$$ds^2_{HT} = -\mathcal{F} \, dt^2 + \frac{1}{\mathcal{F}} \, dr^2 + \mathcal{G} \, r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) , \tag{8}$$

where

$$\mathcal{F} = \left(1 - \frac{2M}{r}\right) \left[1 + \frac{5Q}{4M^3} \, \mathcal{Q}^2_2 \left(\frac{r}{M} - 1\right) P_2(\cos\theta)\right] . \tag{9}$$

We show that

$$r_0 = 3M + \frac{5Q}{16M^2}\,(52 - 45\ln 3)\,. \tag{10}$$

$$\frac{d\phi}{dt} = \omega_\pm = \pm\frac{1}{3\sqrt{3}\,M}\,\left[1 - \frac{5Q}{16M^3}\,(-16 + 15\ln 3)\right]. \tag{11}$$

The quasinormal modes are

$$\omega_{QNM} = \omega^0_{HT} + i\Gamma_{HT} = \pm\, j\omega_\pm + i\gamma_{HT}\left(n + \frac{1}{2}\right), \qquad n = 0, 1, 2, 3, \cdots, \tag{12}$$

where

$$\omega_\pm = \pm\frac{1 - \frac{1}{2}\hat{q}\,(-16 + 15\ln 3)}{3\sqrt{3}\,M}\,, \qquad \gamma_{HT} = \frac{1 + \hat{q}\,(-16 + 15\ln 3)}{3\sqrt{3}M}\,, \qquad \hat{q} := \frac{5Q}{8M^3}\,. \tag{13}$$

I. CONCLUSION

To linear order in Q and second order in J, the QNMs of one such system, namely, the stationary exterior Hartle-Thorne spacetime have been analytically calculated in the eikonal limit using the light-ring method. Our results may be relevant for considerations related to static quadrupolar perturbations generated by *tidal* interactions [5, 6].

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