Exact solution of quantum super-integrable systems with arbitrary spin

Z. Alizadeh, H. Panahi

Department of Physics , University of Guilan , Rasht

Abstract

In this work, we study the super-integrable systems in two-dimensional Euclidean space for particles with arbitrary spin. We show a new connection between deformed oscillator algebra and super-integrable systems with spin in the general form.

Super-integrable systems of quantum mechanics are very interesting and favorite subjects for many physicists and mathematicians. Super-integrability is a guiding sign in search for exactly solvable problems. There exists an interesting class of the exactly solvable systems which were studied by Pronko and Stroganov (ps) [1]. This model is a prefect example of the class of super-integrable systems, including neutral particles with non-trivial spin and dipole moment interacting with an external magnetic field. The related quantum mechanical system, (PS) model, includes a magnetic dipole with spin $\frac{1}{2}$ (neutron) moving in the field of a straight line current. This 2d system is maximally super-integrable, since it admits three integrals of motion [1]. We will recount and restrict ourselves to discussion of super-integrable systems with spin and generalize the PS model to the case of arbitrary spin and show that these quantum super-integrable systems can be described in terms of a deformed oscillator algebra [2,3,4]. The very existence of integrals of motion for super-integrable systems presents powerful tools for finding their spectrum energy of system by solving two equations which are determined directly by equations satisfied by the structure function [5]. The Hamiltonian for this super-integrable model of arbitrary spin can be represented in the form $\mathrm{H} = p_x^2 + p_y^2 + \frac{\mu(\mathbf{s},\mathbf{n})}{r}$, where $\mu(\mathbf{s},\mathbf{n})$ is a $(2s+1) \times (2s+1)$ dimensional matrix depending on $\mathbf{n} = (n_x, n_y)$, $n_x = \frac{x}{r}$, $n_y = \frac{y}{r}$. For each quantum super-integrable system there exists a quantum algebra and one can make a quantum algebra by using constants of motion, which is compatible with the deformed oscillator algebra as [2,4]

$$\mathcal{N} = \mathcal{N}(\mathrm{H}, A_1, A_2), \qquad \mathcal{N}^\dagger = \mathcal{N}, \qquad \mathcal{A} = \mathcal{A}(\mathrm{H}, A_1, A_2),$$

$$[\mathcal{N}, \mathcal{A}] = -\mathcal{A}, \qquad [\mathcal{N}, \mathcal{A}^\dagger] = \mathcal{A}^\dagger, \qquad [\mathcal{A}^\dagger\mathcal{A}, \mathcal{A}\mathcal{A}^\dagger] = 0, \tag{1}$$

$$\mathcal{N} \mid \mathrm{E}, \mathrm{n}\rangle = \mathrm{n} \mid \mathrm{E}, \mathrm{n}\rangle, \quad \mathcal{A}\mathcal{A}^\dagger = \Phi(\mathrm{H}, \mathcal{N}+1), \quad \mathcal{A}^\dagger \mathcal{A} = \Phi(\mathrm{H}, \mathcal{N}),$$

where Φ(E,n), named as structure function, is a real positive definite function and satisfies the following conditions:

$$\Phi(\mathrm{E}, \mathrm{n}) = 0, \qquad \text{for} \qquad \mathrm{n} = 0 \quad \text{and} \quad \mathrm{n} = \mathrm{N}+1$$

$$\Phi(\mathrm{E}, \mathrm{n}) > 0, \qquad \text{for} \qquad \mathrm{n} = 1,2,3,\ldots . \tag{2}$$

Indeed, if the relations (1) and (2) are satisfied, then the system possesses the structure of the deformed oscillator algebra, and we can therefore calculate the spectra of system from the Fock space of creation and annihilation operators. After some conditions, the general form of the Hamiltonian for the quantum system which describes neutral particle with arbitrary spin **s** in the field of rectilinear electric current is equation:

$$\mathcal{H}_s = p_x^2 + p_y^2 + \frac{\mu(\mathbf{s},\mathbf{n})}{r}, \tag{3}$$

where $p_x = -i\frac{\partial}{\partial x}$, $p_y = -i\frac{\partial}{\partial y}$, $\mathbf{r} = x\hat{\imath} + y\hat{\jmath}$ and $\mathbf{s} = (s_x, s_y, s_z)$ is the spin operator.

According to ref. [6], for super-integrability of the above Hamiltonian, it must have two additional integrals of motion

$$A_x = \frac{1}{2}(p_x J_z + J_z p_x) - \frac{y}{2r}\mu(\boldsymbol{s},\boldsymbol{n}), \tag{4}$$

$$A_y = \frac{1}{2}(p_y J_z + J_z p_y) + \frac{x}{2r}\mu(\boldsymbol{s},\boldsymbol{n}).$$

Now, using the above idea, we define the deformed oscillator algebra by the following operators for the system defined by the Hamiltonian (3):

$$\mathcal{N} = J_z + u1, \qquad \mathcal{A}^\dagger = A_x + iA_y, \qquad \mathcal{A} = A_x - i A_y, \qquad [A_x, A_y] = -iJ_z\mathcal{H}_s \tag{5}$$

where satisfy the following deformed oscillator algebra:

$$[\mathcal{N}, \mathcal{A}] = -\mathcal{A}, \qquad [\mathcal{N}, \mathcal{A}^\dagger] = \mathcal{A}^\dagger, \qquad [\mathcal{A}^\dagger, \mathcal{A}] = -2J_z\mathcal{H}_s. \tag{6}$$

Therefore we can consider a direct sum of Fock spaces as the corresponding space of solutions and we name it as matrix Fock space if

$$\Phi(\acute{E}, n)I = 0, \quad \text{for} \quad n = 0 \text{ and } n = N + 1 \tag{7}$$

$$\Phi(\acute{E}, n)I > 0, \quad \text{for} \quad n = 1,2,\ldots$$

where É is now the energy of the spectrum of system and I is the identity matrix for matrix representation of the Fock space.

The structure function is also given by

$$\Phi I = \mathcal{A}^\dagger \mathcal{A}\, I = A_x^2 + A_y^2 - i\,[A_x, A_y], \tag{8}$$

and the relation between constants of motion as the Casimir relation can be obtained as

$$(A_x^2 + A_y^2)I = J_z^2\mathcal{H}_s + \frac{\mathcal{H}_s}{4} + \frac{\mu^2}{4}, \tag{9}$$

where, by inserting it in the structure function, we get

$$\Phi(\mathcal{H}_s,\mathcal{N})I = (\mathcal{N} - u)^2\mathcal{H}_s + \frac{\mathcal{H}_s}{4} - (\mathcal{N} - u)\mathcal{H}_s + \frac{\mu^2}{4}. \tag{10}$$

From now on, we can write the Hamiltonian in (3) in terms of the structure function as

$$\mathcal{H}_s = \frac{\Phi(\mathcal{H}_s,\mathcal{N})\mathrm{I}-\frac{\mu^2}{4}}{\left(\mathcal{N}-u-\frac{1}{2}\right)^2}. \quad (11)$$

Now, according to the eigenvalue equation $(\mathcal{H}_s - \acute{\mathrm{E}}\mathrm{I})\psi = 0$, we should solve the following equation for finding the non-zero solution:

$$\det(\mathcal{H}_s - \acute{\mathrm{E}}\mathrm{I}) = 0. \quad (12)$$

It is also obvious that we must use the Fock space representation by using of the conditions of (7), that is the structure function should be zero for n = 0 and n = N + 1, and so, by solving the obtained equation, we can determine the constant of u and also the energy spectrum of the quantum maximally super-integrable system.

Hence as application example, let us regard a special class of Schrodinger-Pauli equations describing neutral fermions with non-trivial dipole momentum interacting with the external magnetic field where $\mu(\boldsymbol{s},\boldsymbol{n})$ in Eq (3) for this model is the magnetic moment of the particle and is defined as

$$\mu(\boldsymbol{s},\boldsymbol{n}) = \lambda\left(s_x n_y - s_y n_x\right), \quad (13)$$

where the coefficient λ is collected by all constants.

Terefore, the final form of the PS model, based on the Schrodinger-Pauli Hamiltonian, will be

$$\mathcal{H}_s = p_x^2 + p_y^2 + \lambda\,\frac{y s_x - x s_y}{r^2}, \quad (14)$$

Now, let us take the matrix μ(**s**,**n**) according to eq. (13), and obtain the relation of μ^2 as

$$\mu^2 = \frac{\lambda^2}{4}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \quad (15)$$

hence, the form of $\mathcal{H}_s$ in (11) will become

$$\mathcal{H}_s = \frac{\left(\Phi-\frac{\lambda^2}{16}\right)I}{\left(\mathcal{N}-u-\frac{1}{2}\right)^2}. \quad (16)$$

According to conditions (7), the structure function should be zero for n = 0 and n = N + 1, which, using eq. (12), easily gives the unknown parameter u, and then we calculate the degenerate spectrum of energy as

$$u = \frac{N}{2}, \quad \acute{E} = \frac{-\lambda^2}{4(N+1)^2}. \quad (17)$$

Other applicable examples for spin 1 and $\frac{3}{2}$ can be seen in Ref. [5, 6].

Conclusion

In this paper, we have studied the interesting class of systems in 2-dimensional Euclidean space, which have more than two integrals of motion. The presented systems have neutral particles with arbitrary spin in the magnetic field. We have shown that these quantum maximally super-integrable systems with spin can be described in terms of a deformed oscillator algebra. We have calculated the spectra of the system by solving two equations obtained from the structure function [5].

References

1. G.P. Pronko, Y.G. Stroganov, *Sov. Phys. JETP* **45**, 1075 (1977).
2. C. Daskaloyannis, *J. Phys. A* **24**, L789 (1991).
3. H. Panahi, Z. Alizadeh, *Chin. Phys. B* **22**, 060304 (2013).
4. D. Bonatsos, C. Daskaloyannis, K. Kokkotas, *Phys. Rev. A* **50**, 3700 (1994).
5. Z. Alizadeh, H. Panahi ,*Eur. Phys. J. Plus* **129** (2014)
6. G.P. Pronko, *J. Phys. A: Math. Theor.* **40**, 13331 (2007).